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Fractional Divided Differences and the Solution of Differential Equations of Fractional Order

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1. INTRODUCTION

In our previous papers [1–6, 14] we have developed a theory of divided differences and rational functions that yields general methods for the solution of linear differential-like equations. Our methods are based on linear algebra constructions and are an alternative to the classical methods based on integral transforms or operational calculi.

In this paper we extend our theory by introducing divided differences of fractional order and generalized exponential polynomials. These functions are closely related to Mittag–Leffler functions and are the solutions of homogeneous differential equations associated with a fractional differential operator related to Weyl's operator. We also apply our methods (Section 4) to find solutions of a Cauchy bounded problem for the Riemann–Liouville fractional differential operator that was studied by Luchko and Srivastava [10] using a Mikusiński-type operational calculus. Finally, (Section 5) we solve some examples of the Laplace equation that were studied by Yoshida [8].

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In Section 2 we present briefly the main results about rational functions and divided differences that we use in our development. In Section 3 we use contour integrals, associated with Hankel's integral representation for the reciprocal of the Gamma function, to define generalized divided differences, a fractional differentiation operator L_μ, and a family of functions that form a basis for a vector space $\mathscr{G}$ of generalized exponential polynomials. Using a general result from [14] we find an explicit form for the general solution of equations of the form $u(L_\mu)g = f$, where u is a polynomial and f is a given element of $\mathscr{G}$.

In Section 4 we describe the relationships of our generalized exponential polynomials with functions of Mittag–Leffler type and show that our operator L_μ coincides with the usual Riemann–Liouville fractional differential operator on certain subspace of $\mathscr{G}$. We also find the solution of certain Cauchy boundary problems.

In Section 5 we use our results to find solutions of some examples of the Laplace differential equation and show that our methods can be used to solve Bessel, confluent hypergeometric, and Laguerre equations.

The Hopf algebra aspects of our approach, and some results about convolutions and inner products, are discussed in [6]. See also, [3, 4, and 5].

2. PROPER RATIONAL FUNCTIONS AND DIVIDED DIFFERENCES

In this section we present some notions about rational functions and divided difference functionals that will be used in our development.

We denote by $\mathscr{P}$ the complex vector space of all polynomials in the indeterminate z, and by $\mathscr{P}^*$ the dual vector space of $\mathscr{P}$. For $n \geq 0$ the subspace of $\mathscr{P}$ of all polynomials whose degree is at most n is denoted by $\mathscr{P}_n$, and its dual by $\mathscr{P}_n^*$. The duality pairing of $\mathscr{P}^*$ and $\mathscr{P}$ is described with the angular bracket notation, that is,

$$\langle L, p \rangle = Lp, \qquad L \in \mathscr{P}^*,\ p \in \mathscr{P}.$$

The *basic Taylor functionals* $T_{a,k}$ are the elements of $\mathscr{P}^*$ defined by

$$\langle T_{a,k}, p \rangle = \frac{1}{k!} D^k p(a), \qquad a \in \mathbb{C},\ k \in \mathbb{N},\ p \in \mathscr{P},$$

where D denotes the usual differential operator. Observe that Taylor functionals may be applied to functions more general than polynomials, like entire or meromorphic functions. The vector space generated by the basic Taylor functionals is denoted by $\mathscr{T}$, and its elements are called Taylor functionals.

Since evaluations are multiplicative functionals, by Leibniz's rule for differentiation we obtain

$$\langle T_{a,k}, fg \rangle = \sum_{j=0}^{k} \langle T_{a,j}, f \rangle \langle T_{a,k-j}, g \rangle$$

for any functions f and g that are sufficiently differentiable at a. This formula is called Leibniz's rule for Taylor functionals.

The space $\mathscr{R}$ of the proper rational functions is the space generated by the basic rational functions

$$r_{a,k}(t) = \frac{1}{(t-a)^{1+k}}, \qquad a \in \mathbb{C}, k \in \mathbb{N}.$$

Let $r \geq 0$, let $a_0, a_1, \ldots, a_r$ be distinct complex numbers, and let m_0, $m_1, \ldots, m_r$ be positive integers. Define $n + 1 = \Sigma m_i$ and

$$u(z) = \prod_{i=0}^{r} (z - a_i)^{m_i} = z^{n+1} + b_1 z^n + \cdots + b_{n+1}. \tag{2.1}$$

Let $\mathscr{I} = \{(i,j) : 0 \leq i \leq r, 0 \leq j \leq m_i - 1\}$ and define the polynomials

$$q_{i,j}(z) = \frac{u(z)}{(z-a_i)^{m_i - j}}, \qquad (i,j) \in \mathscr{I}, \tag{2.2}$$

and the linear functions $L_{i,j}$ by

$$\langle L_{i,j}, p \rangle = \left\langle T_{a_i,j}, \frac{p}{q_{i,0}} \right\rangle, \qquad p \in \mathscr{P}, (i,j) \in \mathscr{I}. \tag{2.3}$$

Using Leibniz's rule for Taylor functionals it is easy to see that

$$\langle L_{i,j}, q_{k,s} \rangle = \delta_{(i,j),(k,s)}, \qquad (i,j),(k,s) \in \mathscr{I}. \tag{2.4}$$

Consequently the polynomials $q_{i,j}$ form a basis for $\mathscr{P}_n$ and the functionals $L_{i,j}$ form a basis for the space $\mathscr{P}_n^*$. So, for every polynomial p in $\mathscr{P}_n$ we have

$$p(z) = \sum_{(i,j) \in \mathscr{I}} \langle L_{i,j}, p \rangle q_{i,j}(z). \tag{2.5}$$

Dividing by $u(z)$ and using (2.2) we get the *partial fractions decomposition formula* (PFD)

$$\frac{p(z)}{u(z)} = \sum_{(i,j) \in \mathscr{I}} \frac{\langle L_{i,m_i-1-j}, p \rangle}{(z-a_i)^{1+j}}, \qquad p \in \mathscr{P}_n. \tag{2.6}$$

Taking $u(z) = (z-a)^{1+k}(z-b)^{1+m}$, with $a \neq b$, $p(z) = 1$, and applying the PFD (2.6), we obtain the multiplication formula

$$r_{a,k} r_{b,m} = \sum_{j=0}^{k} C(a,j;b,m) r_{a,k-j} + \sum_{j=0}^{m} C(b,j;a,k) r_{b,m-j}, \quad (2.7)$$

where the coefficients are defined by

$$C(a,j;b,i) = (-1)^j \binom{j+i}{j} (a-b)^{-1-j-i}, \qquad a \neq b,\, i,j \in \mathbb{N}. \quad (2.8)$$

Note that $r_{a,k} r_{a,m} = r_{a,1+k+m}$.

Taking $p = 1$ in the PFD (2.6), we get

$$\frac{1}{u(z)} = \sum_{i=0}^{r} \sum_{j=0}^{m_i - 1} \alpha_{i,j} r_{a_i,j}(z), \quad (2.9)$$

where

$$\alpha_{i,j} = \langle L_{i,m_i-1-j}, 1 \rangle. \quad (2.10)$$

Note that the coefficients $\alpha_{i,j}$ depend only on the roots of $u(z)$ and their multiplicities. For more details see [3] and [4].

The spaces $\mathscr{R}$ and $\mathscr{T}$ are isomorphic as vector spaces. We have the natural isomorphism that sends the basic rational function $r_{a,k}$ to $T_{a,k}$. For $p \in \mathscr{P}_n$, the element of $\mathscr{T}$ that corresponds to p/u under this isomorphism is the functional

$$A = A(p/u) = \sum_{(i,k) \in \mathscr{I}} \langle L_{i,k}, p \rangle T_{a_i, m_i - 1 - k}.$$

We say that a complex valued function f of a complex variable t is defined on the roots of u if and only if $\langle T_{a_i,k}, f \rangle$ is well defined for (i,k) in $\mathscr{I}$. For any such f we have

$$\langle A, f \rangle = \sum_{(i,k) \in \mathscr{I}} \langle L_{i,k}, p \rangle \langle T_{a_i, m_i-1-k}, f \rangle.$$

From now on we write $\langle p/u, f \rangle$ instead of $\langle A, f \rangle$. That is, we identify p/u with its image in $\mathscr{T}$ under the natural isomorphism.

The definition of $L_{i,k}$ and Leibniz's rule give us

$$\left\langle \frac{p}{u}, f \right\rangle = \sum_{i=0}^{r} \langle L_{i,m_i-1}, pf \rangle,$$

which can be written as

$$\left\langle \frac{p}{u}, f \right\rangle = \sum_{i=0}^{r} \text{Residue at } a_i \text{ of } \frac{pf}{u}, \qquad p \in \mathscr{P}_n. \tag{2.11}$$

The linear functional associated with $1/u$ is called the *divided difference functional* with respect to the roots of u. For any f defined on the roots of u we have

$$\left\langle \frac{1}{u}, f \right\rangle = \sum_{i=0}^{r} \text{Residue at } a_i \text{ of } \frac{f}{u}.$$

We list next some elementary properties of divided differences. Most of them are direct consequences of (2.11). For the proofs see [2] or [3].

If u has simple roots $a_0, a_1, \ldots, a_n$ and f is defined on the a_i's then

$$\left\langle \frac{1}{u}, f \right\rangle = \sum_{i=0}^{n} \frac{f(a_i)}{u'(a_i)}. \tag{2.12}$$

For any monic polynomial u of degree $n+1$ we have

$$\left\langle \frac{1}{u(z)}, z^k \right\rangle = \delta_{n,k}, \qquad 0 \le k \le n, \tag{2.13}$$

and $\langle 1/u, z^k \rangle$ is a polynomial in the roots of u for $k > n$.

If f is any function defined on the roots of u then

$$\left\langle \frac{1}{u}, uf \right\rangle = 0. \tag{2.14}$$

If u and v are monic polynomials of positive degree and f is defined on the roots of uv then

$$\left\langle \frac{1}{uv}, vf \right\rangle = \left\langle \frac{1}{u}, f \right\rangle. \tag{2.15}$$

If u and v have no common roots and f is defined on the roots of uv then

$$\left\langle \frac{1}{uv}, f \right\rangle = \left\langle \frac{1}{u}, \frac{f}{v} \right\rangle + \left\langle \frac{1}{v}, \frac{f}{u} \right\rangle. \tag{2.16}$$

If u has degree $n+1$ and f is defined on the roots of u then

$$\left\langle \frac{p}{u}, f \right\rangle = \left\langle \frac{1}{u}, pf \right\rangle, \qquad p \in \mathscr{P}_n. \tag{2.17}$$

If p/u is in $\mathscr{R}$, q is a polynomial, and f is defined on the roots of u, then

$$\left\langle \frac{p}{u}, qf \right\rangle = \left\langle \frac{1}{u}, pqf \right\rangle = \left\langle \frac{1}{u}, rf \right\rangle = \left\langle \frac{r}{u}, f \right\rangle, \tag{2.18}$$

where $pq = vu + r$, and the degree of r is strictly less than the degree of u. That is, r is the residue of pq modulo u.

For any $a \in \mathbb{C}$ and $k \in \mathbb{N}$, we have

$$\left\langle \frac{1}{(z-a)^{1+k}}, e^{zt} \right\rangle = \frac{t^k}{k!} e^{at}. \tag{2.19}$$

We denote by $\mathscr{E}$ the complex vector space generated by the functions given in (2.19). The elements of $\mathscr{E}$ are called *quasi-polynomials or exponential polynomials* and they are the solutions of linear homogeneous differential equations with constant coefficients (see [4]).

The Horner polynomials associated with $u(z)$, denoted by $u_k(z)$, are defined as follows: $u_0 = 1$ and

$$u_k(z) = z^k + b_1 z^{k-1} + \cdots + b_k, \qquad k \geq 1, \tag{2.20}$$

where the b_j are the coefficients of u introduced in (2.1). They satisfy the biorthogonality property

$$\left\langle \frac{z^k u_{n-j}(z)}{u(z)}, 1 \right\rangle = \delta_{j,k}, \qquad 0 \leq k \leq n, 0 \leq j \leq n. \tag{2.21}$$

3. GENERALIZED DIVIDED DIFFERENCES AND EXPONENTIAL POLYNOMIALS

In this section we will generalize (2.19) and define a new space $\mathscr{G}$ of functions as an extension of the space $\mathscr{E}$ of quasi-polynomials. After this we will give a class of functional equations whose solutions are precisely the elements of $\mathscr{G}$. First note that from (2.11) we can write

$$\left\langle \frac{p}{u}, f \right\rangle = \frac{1}{2\pi i} \int_\gamma \frac{p(z) f(z)}{u(z)}\, dz, \tag{3.1}$$

where γ is a simple closed rectifiable positively oriented curve (scroc, see [7, pp. 172, 241]) which encloses $a_0, a_1, \ldots, a_n$.

Using a representation for the reciprocal of the Gamma function by a contour integral due to H. Hankel (see [7, p. 234; 15, p. 324]), we obtain:

$$\left\langle \frac{1}{(z-a)^{1+\mu}}, e^{zt} \right\rangle = \frac{t^{\mu}}{\Gamma(1+\mu)} e^{at}, \qquad a, \mu \in \mathbb{C}. \tag{3.2}$$

If f is an analytic function on a set $U \subset \mathbb{C}$ and h is an analytic function on U except for isolated singularities $z_0, z_1, \dots, z_n$, we define

$$\langle h(z), f(z) \rangle = \frac{1}{2\pi i} \int_{\gamma} h(z) f(z)\, dz, \tag{3.3}$$

where γ is a scroc in U such that $z_0, z_1, \dots, z_n$ lie inside it. This expression generalizes (2.19).

It is easy to show that

$$\langle h(z), f'(z) \rangle = \langle -h'(z), f(z) \rangle \tag{3.4}$$

and that properties analogous to (2.12)–(2.18) hold.

PROPOSITION 3.1. *Let β and t be arbitrary complex numbers. Then*

$$\langle (1+z)^{\beta}, e^{zt} \rangle = \sum_{k=0}^{\infty} \binom{\beta}{k} \frac{t^{k-\beta-1}}{\Gamma(k-\beta)}. \tag{3.5}$$

Proof. Let α and z be complex numbers with $|z| < 1$. By the binomial theorem (see [8, p. 39]) we have

$$(1+z)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}, \qquad k \ge 1,$$

and $\binom{\alpha}{0} = 1$. Then, for complex numbers β and z, with $|z| > 1$, we have

$$(1+z)^{\beta} = z^{\beta}\left(1+\frac{1}{z}\right)^{\beta} = z^{\beta} \sum_{k=0}^{\infty} \binom{\beta}{k} \frac{1}{z^k} = \sum_{k=0}^{\infty} \binom{\beta}{k} \frac{1}{z^{k-\beta}}.$$

Therefore, using (3.2) we obtain

$$\begin{aligned}\left\langle (1+z)^{\beta}, e^{zt}\right\rangle &= \left\langle \sum_{k=0}^{\infty} \binom{\beta}{k} \frac{1}{z^{k-\beta}}, e^{zt}\right\rangle \\ &= \sum_{k=0}^{\infty} \binom{\beta}{k} \left\langle \frac{1}{z^{k-\beta}}, e^{zt}\right\rangle \\ &= \sum_{k=0}^{\infty} \binom{\beta}{k} \frac{t^{k-\beta-1}}{\Gamma(k-\beta)}.\end{aligned}$$

∎

There is another case of interest for us. Let z_1 and z_2 be two nonzero distinct complex numbers. Let λ, δ, and z be elements of $\mathbb{C}$ such that $|z| > \max\{|z_1|, |z_2|\}$. Then

$$(z-z_1)^{\lambda}(z-z_2)^{\delta} = \sum_{k=0}^{\infty} (-1)^k c_k \frac{1}{z^{k-\lambda-\delta}}, \tag{3.6}$$

where

$$c_k = \sum_{j=0}^{k} \binom{\lambda}{j} \binom{\delta}{k-j} z_1^j z_2^{k-j}. \tag{3.7}$$

Consequently we have the following result.

PROPOSITION 3.2. *For $z, z_1, z_2, \lambda, \delta$ as in the previous paragraph we have*

$$\left\langle (z-z_1)^{\lambda}(z-z_2)^{\delta}, e^{zt}\right\rangle = \sum_{k=0}^{\infty} (-1)^k c_k \frac{t^{k-\lambda-\delta-1}}{\Gamma(k-\lambda-\delta)}. \tag{3.8}$$

Note that if $z_1 z_2 = 0$ or $z_1 = z_2$, then we can use (3.5) to obtain a series representation for the expression in (3.8).

We define next a family of functions which generalizes our previous results and contains the exponential polynomials.

Let μ be a positive real number. For a in $\mathbb{C}$, λ in $\mathbb{R}$, and k in $\mathbb{N}$ we define the functions

$$G_{\lambda}(a,t) = \left\langle \frac{z^{\lambda}}{z^{\mu}-a}, e^{zt}\right\rangle \tag{3.9}$$

and

$$g_{\lambda,a,k}(t) = T_{a,k} G_{\lambda}(x,t), \tag{3.10}$$

where the Taylor functional acts with respect to x, and in this case the path of integration is a loop which starts and ends at $-\infty$ and encircles the disc $|z| \le |a|^{1/\mu}$ in the positive sense: $-\pi \le \arg t \le \pi$ on $\mathbb{C}$. This is called a Hankel integration path.

We denote by $\mathscr{G}$ the complex vector space generated by the functions $g_{j,a,k}$, for (j, a, k) in $\mathbb{N} \times \mathbb{C} \times \mathbb{N}$, and for each fixed integer j_0 we denote by $\mathscr{G}_{j_0}$ the subspace generated by the functions $g_{j_0, a, k}$, for (a, k) in $\mathbb{C} \times \mathbb{N}$.

Note that

$$g_{\lambda, a, k}(t) = \left\langle \frac{z^{\lambda}}{(z^{\mu} - a)^{1+k}}, e^{zt} \right\rangle. \tag{3.10.a}$$

In particular, if $\lambda = 0$ and $\mu = 1$ then

$$g_{0, a, k}(t) = \left\langle \frac{1}{(z - a)^{1+k}}, e^{zt} \right\rangle = \frac{t^k}{k!} e^{at}.$$

That is why we say that the elements of $\mathscr{G}$ are *generalized exponential polynomials*.

If $\lambda < \mu$ and $\mu > 0$ we will see in the next section that $g_{\lambda, a, k}$ can be expressed in terms of a Mittag–Leffler function. If $\lambda \ge \mu$ then $g_{\lambda, a, k}$ contains additionally a linear combination of fractional powers of t with exponents that are less than or equal to -1.

We conclude this section by introducing a class of functional equations whose space of solutions is $\mathscr{G}$. For $\alpha > 0$ we define the linear operator L_α on $\mathscr{G}$ by

$$L_\alpha g_{j, a, k}(t) = \left\langle \frac{z^{j}}{(z^{\mu} - a)^{1+k}}, z^{\alpha} e^{zt} \right\rangle. \tag{3.11}$$

We shall study the solutions of equations of the form

$$u(L_\mu) g(t) = f(t), \tag{3.12}$$

where g is an unknown function, u is a nonzero polynomial as in (2.1), t is a real or complex variable, and f is an element of $\mathscr{G}$.

We show first that the operator L_μ is a linear map of the space $\mathscr{G}$ into itself, that is, L_μ: $\mathscr{G} \to \mathscr{G}$. If $k = 0$ we have

$$L_\mu g_{j, a, 0}(t) = \left\langle \frac{z^{j+\mu}}{z^{\mu} - a}, e^{zt} \right\rangle = \left\langle z^j + \frac{a z^j}{z^{\mu} - a}, e^{zt} \right\rangle = a g_{j, a, 0}(t),$$

since $\langle z^j, e^{zt}\rangle = 0$ for all j in $\mathbb{N}$. If $k \geq 1$, then

$$\begin{aligned} L_\mu g_{j,a,k}(t) &= \left\langle \frac{z^{j+\mu}}{(z^\mu - a)^{1+k}}, e^{zt} \right\rangle \\ &= \left\langle \frac{az^j}{(z^\mu - a)^{1+k}} + \frac{z^j}{(z^\mu - a)^k}, e^{zt} \right\rangle \\ &= ag_{j,a,k}(t) + g_{j,a,k-1}(t). \end{aligned}$$

Hence the operator L_μ has the property (see [14, Eq. (4.3)])

$$L_\mu g_{j,a,k} = \begin{cases} ag_{j,a,0}, & \text{if } k = 0, \\ ag_{j,a,k} + g_{j,a,k-1}, & \text{if } k \geq 1. \end{cases} \tag{3.13}$$

In a similar way and using (3.2) we get the following more general relations, which hold for j, q in $\mathbb{Z}$ and $k \geq 1$.

$$L_{\mu-q-1} g_{j,a,k} = \begin{cases} ag_{j-q-1,a,0}, & \text{if } k = 0,\ j \geq q+1, \\ \dfrac{t^{q-j}}{\Gamma(q-j+1)} + ag_{j-q-1,a,0}, & \text{if } k = 0,\ j \leq q, \\ ag_{j-q-1,a,k} + g_{j-q-1,a,k-1}, & \text{if } k \geq 1. \end{cases} \tag{3.13.a}$$

We define the commutative convolution product $*$ on $\mathscr{G}$ as follows. Let i, j, k, and m be in $\mathbb{N}$ and let a, b be complex numbers. If $a \neq b$ we define

$$g_{i,a,k} * g_{j,b,m} = \sum_{l=0}^{k} C(a,l;b,m) g_{i+j,a,k-l} + \sum_{l=0}^{m} C(b,l;a,k) g_{i+j,b,m-l}, \tag{3.14}$$

where the coefficient functions are defined in (2.8), and

$$g_{i,a,k} * g_{j,a,m} = g_{i+j,a,1+k+m}. \tag{3.15}$$

For any a in $\mathbb{C}$ it is obvious that

$$(L_\mu - aI) g_{j,a,k} = \begin{cases} 0, & \text{if } k = 0, \\ g_{j,a,k-1}, & \text{if } k \geq 1, \end{cases} \tag{3.16}$$

and thus

$$\left(L_\mu - aI\right)^{1+m} g_{j,a,k} = 0, \qquad m \geq k. \tag{3.17}$$

By induction on m it is easy to see that

$$\left(L_\mu - aI\right)^m g_{j,b,k} = \sum_{i=0}^{s} \binom{m}{i} (b-a)^{m-i} g_{j,b,k-i}, \tag{3.18}$$

where $s = \min\{m, k\}$. Therefore, for $a \neq b$ we have

$$\left(L_\mu - aI\right)^m g_{j,b,k} \neq 0, \qquad j, k, m \in \mathbb{N}. \tag{3.19}$$

A straightforward computation yields

$$L_\mu(g_{0,a,k} * g_{j,b,m}) = \left(L_\mu g_{0,a,k}\right) * g_{j,b,m} + g_{j,b,m} \Phi g_{0,a,k}, \tag{3.20}$$

where Φ is the linear functional on $\mathscr{G}$ defined by $\Phi g_{j,a,k} = \delta_{0,k}$. By linearity we obtain

$$L_\mu(g * f) = \left(L_\mu g\right) * f + f \Phi g, \qquad f \in \mathscr{G}, \ g \in \mathscr{G}_0. \tag{3.21}$$

Using induction it is easy to prove that

$$\left(L_\mu - aI\right)^{1+k} (g_{0,a,k} * f) = f, \qquad (a,k) \in \mathbb{C} \times \mathbb{N}, f \in \mathscr{G}. \tag{3.22}$$

Let $u(z)$ be a monic polynomial of positive degree as in (2.1). Then, by (2.9), the PFD for $1/u$ is

$$\frac{1}{u(z)} = \sum_{i=0}^{r} \sum_{k=0}^{m_i - 1} \alpha_{i,k} r_{a_i,k}(z).$$

We define

$$h_u = \sum_{i=0}^{r} \sum_{k=0}^{m_i - 1} \alpha_{i,k} g_{0,a_i,k}. \tag{3.23}$$

As a particular case of [14, Theorem 4.1] we obtain

THEOREM 3.1. *Let f be a given element of $\mathscr{G}$ and let the polynomial u be as above. Then the general solution of the equation $u(L_\mu)g = f$ is*

$$g = h_u * f + \tilde{g}, \tag{3.24}$$

where h_u is defined in (3.23) *and $\tilde{g}$ is any element of the subspace of $\mathscr{G}$ generated by $\{g_{j,a_i,k} : j \in \mathbb{N}, 0 \leq i \leq r, 0 \leq k \leq m_i - 1\}$.*

4. AN APPLICATION TO FRACTIONAL DIFFERENTIAL EQUATIONS

Let μ be a positive real number. The operator I_μ defined by

$$(I_\mu f)(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)\, dt \tag{4.1}$$

is called the *Riemann–Liouville fractional integral operator* of order μ.

The *Riemann–Liouville fractional differential operator* of order μ, denoted by D_μ, is defined as

$$(D_\mu f)(x) = \left(\frac{d}{dx}\right)^{\eta} (I_{\eta-\mu} f)(x), \qquad \eta = \begin{cases} [\mu]+1, & \text{if } \mu \notin \mathbb{N}, \\ \mu, & \text{if } \mu \in \mathbb{N} \end{cases}. \tag{4.2}$$

For the theory and applications of these operators one may refer to [11–13].

In this section we study the solutions of fractional differential equations of the form

$$u(D_\mu) g(t) = f(t), \tag{4.3}$$

where u is a nonzero polynomial given in (2.1), t is a real or complex variable, and we denote by D_μ^k the composition of k Riemann–Liouville fractional differential operators.

A two-parameter *function of the Mittag–Leffler type* is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta \in \mathbb{R}^+, \tag{4.4}$$

or equivalently by the integral

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{t^{\alpha-\beta} e^t}{t^\alpha - z}\, dt, \tag{4.5}$$

where the path of integration $\mathscr{C}$ is a Hankel path as in (3.9) and (3.10).

The following two propositions relate our results of the previous section with the above concepts.

PROPOSITION 4.1. *Let μ and λ be real numbers such that $\mu > 0$ and $\lambda < \mu$. Then*

$$g_{\lambda,a,k}(t) = \frac{t^{\mu(k+1)-\lambda-1}}{k!} E^{(k)}_{\mu,\mu-\lambda}(a t^\mu), \tag{4.6}$$

where

$$E^{(k)}_{\alpha,\beta}(y) = \frac{d^k}{dy^k} E_{\alpha,\beta}(y) = \sum_{j=0}^{\infty} \frac{(j+k)!\,y^j}{j!\,\Gamma(\alpha j + \alpha k + \beta)}.$$

Proof. For z such that $|z| > |a|^{1/\mu}$, we have

$$\frac{1}{z^\mu - a} = \frac{1}{z^\mu(1 - az^{-\mu})} = \frac{1}{z^\mu} \sum_{k=0}^{\infty} \left(\frac{a}{z^\mu}\right)^k,$$

so that

$$\frac{z^\lambda}{z^\mu - a} = \sum_{k=0}^{\infty} \frac{a^k}{z^{\mu k + \mu - \lambda}},$$

and using (3.9) and (3.2) we obtain

$$\begin{aligned} G_\lambda(a,t) &= \left\langle \frac{z^\lambda}{z^\mu - a}, e^{zt} \right\rangle \\ &= \sum_{k=0}^{\infty} \left\langle \frac{a^k}{z^{\mu k + \mu - \lambda}}, e^{zt} \right\rangle \\ &= \sum_{k=0}^{\infty} a^k \frac{t^{\mu k + \mu - \lambda - 1}}{\Gamma(\mu k + \mu - \lambda)} \\ &= t^{\mu - \lambda - 1} \sum_{k=0}^{\infty} \frac{(at^\mu)^k}{\Gamma(\mu k + \mu - \lambda)}, \end{aligned}$$

that is,

$$G_\lambda(a,t) = t^{\mu - \lambda - 1} E_{\mu, \mu - \lambda}(at^\mu). \tag{4.7}$$

From (4.7) and (3.10) we obtain (4.6). ∎

Proposition 4.1 can be written in terms of the functions $E^{\rho}_{\alpha,\beta}$, defined in [10] by

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{k!\,\Gamma(\alpha k + \beta)}, \qquad \rho \in \mathbb{N},$$

due to the relation

$$\frac{1}{\rho!} E^{(\rho)}_{\alpha,\beta}(z) = E^{\rho+1}_{\alpha,\alpha\rho+\beta}(z).$$

Suppose that λ is real and $\lambda \geq \mu$. Let l_0 be the least integer such that $l_0 \mu \leq \lambda < (l_0 + 1)\mu$ and define $\alpha = \lambda - l_0 \mu$. So $\lambda = l_0 \mu + \alpha$, with $0 \leq \alpha < \mu$. Hence

$$\frac{z^{\lambda}}{z^{\mu} - a} = \frac{z^{\alpha}(z^{\mu})^{l_0}}{z^{\mu} - a} = \sum_{l=0}^{l_0 - 1} a^{l_0 - 1 - l} z^{\mu l + \alpha} + \frac{a^{l_0} z^{\alpha}}{z^{\mu} - a},$$

and using (3.2) we get

$$\begin{aligned} G_{\lambda}(a, t) &= \sum_{l=0}^{l_0 - 1} a^{l_0 - 1 - l} \langle z^{\mu l + \alpha}, e^{zt} \rangle + a^{l_0} t^{\mu - \alpha - 1} E_{\mu, \mu - \alpha}(a t^{\mu}) \\ &= \sum_{l=0}^{l_0 - 1} \frac{a^{l_0 - 1 - l} t^{-\mu l - \alpha - 1}}{\Gamma(-\mu l - \alpha)} + a^{l_0} t^{\mu - \alpha - 1} E_{\mu, \mu - \alpha}(a t^{\mu}). \end{aligned}$$

The last expression together with (3.10) shows that the function $g_{\lambda, a, k}$ contains a linear combination of powers of t with exponents that are less than or equal to -1, and in consequence $D_{\mu} g_{\lambda, a, k}$ does not exist if $\lambda \geq \mu$.

A direct calculation from (4.2) and (4.6) shows that the functions $g_{j, a, k}$, with j in $\mathbb{N}$ and $0 \leq j < \mu$, satisfy the following property.

PROPOSITION 4.2. *Let D_{μ} be the Riemann–Liouville fractional differential operator of order μ and let j be in $\mathbb{N}$, with $j < \mu$. Then*

$$D_{\mu} g_{j, a, k} = \begin{cases} a g_{j, a, 0}, & \text{if } k = 0, \\ a g_{j, a, k} + g_{j, a, k-1}, & \text{if } k \geq 1. \end{cases} \tag{4.8}$$

Proof. We have that

$$g_{j, a, k}(t) = \sum_{m=k}^{\infty} \binom{m}{k} \frac{a^{m-k} t^{m\mu + \mu - j - 1}}{\Gamma(m\mu + \mu - j)}.$$

If $k = 0$ then

$$D_{\mu} g_{j, a, 0}(t) = \sum_{m=1}^{\infty} \frac{a^m t^{m\mu - j - 1}}{\Gamma(m\mu - j)} = a \sum_{i=0}^{\infty} \frac{a^i t^{i\mu + \mu - j - 1}}{\Gamma(i\mu + \mu - j)} = a g_{j, a, 0}(t).$$

Now, for $k \geq 1$ we have

$$\begin{aligned}
D_\mu g_{j,a,k}(t) &= \sum_{m=k}^{\infty} \binom{m}{k} \frac{a^{m-k} t^{m\mu-j-1}}{\Gamma(m\mu - j)} \\
&= \sum_{i=k-1}^{\infty} \binom{i+1}{k} \frac{a^{i+1-k} t^{i\mu+\mu-j-1}}{\Gamma(i\mu + \mu - j)} \\
&= \sum_{i=k}^{\infty} \binom{i+1}{k} \frac{a^{i+1-k} t^{i\mu+\mu-j-1}}{\Gamma(i\mu + \mu - j)} + \frac{t^{k\mu-j-1}}{\Gamma(k\mu - j)} \\
&= \sum_{i=k}^{\infty} \binom{i}{k} \frac{a^{i+1-k} t^{i\mu+\mu-j-1}}{\Gamma(i\mu + \mu - j)} \\
&\quad + \sum_{i=k}^{\infty} \binom{i}{k-1} \frac{a^{i+1-k} t^{i\mu+\mu-j-1}}{\Gamma(i\mu + \mu - j)} + \frac{t^{k\mu-j-1}}{\Gamma(k\mu - j)} \\
&= a \sum_{i=k}^{\infty} \binom{i}{k} \frac{a^{i-k} t^{i\mu+\mu-j-1}}{\Gamma(i\mu + \mu - j)} \\
&\quad + \sum_{i=k-1}^{\infty} \binom{i}{k-1} \frac{a^{i-(k-1)} t^{i\mu+\mu-j-1}}{\Gamma(i\mu + \mu - j)} \\
&= a g_{j,a,k}(t) + g_{j,a,k-1}(t).
\end{aligned}$$

∎

Note that it is possible to calculate $D_\mu^i g_{j,a,k}$, for $i \geq 1$, in terms of the functions $g_{j,a,m}$, with $0 \leq m \leq k$, by using (4.8) repeatedly. Let $\mathscr{H}$ be the complex vector space generated by the functions $g_{j,a,k}$, with $(j, a, k) \in \mathbb{N} \times \mathbb{C} \times \mathbb{N}$ and $0 \leq j \leq \eta - 1$. Then $D_\mu: \mathscr{H} \to \mathscr{H}$. Moreover, Proposition 4.2 says that $D_\mu = L_\mu$ on $\mathscr{H}$, where L_μ was defined in (3.11). Therefore, we obtain from Theorem 3.1 the following result.

COROLLARY 4.1. *Let f be a given element of $\mathscr{G}$ and let the polynomial u be as above. Then the general solution of the equation $u(D_\mu)g = f$ is*

$$g = h_u * f + \sum_{j=0}^{\eta-1} \sum_{i=0}^{r} \sum_{k=0}^{m_i-1} \beta_{j,i,k} g_{j,a_i,k}, \tag{4.9}$$

where h_u is defined in (3.23) and the coefficients $\beta_{j,i,k}$ are arbitrary complex numbers.

A direct calculation shows that $D_{\mu-q-1} = L_{\mu-q-1}$ on $\mathscr{H}$ for $q = 0, 1, \dots, \eta - 1$. From this, (4.6), and (3.13.a) we get

$$\lim_{t\to 0} D_{\mu-q-1} g_{j,a,k}(t) = \delta_{(j,k),(q,0)}. \tag{4.10}$$

We will apply our previous results to solve the problem:

$$u(D_\mu)g = f, \qquad \lim_{t\to 0}\left(D_{\mu-q-1}D_\mu^l g\right)(t) = d_{l,q}, \tag{4.11}$$

where $l = 0, 1, \dots, n$; $q = 0, 1, \dots, \eta - 1$; and $f \in \mathscr{H}$. The problem (4.11) is called a *Cauchy boundary-value problem for the Riemann–Liouville fractional differential operator* D_μ. For $\mu = 1$ it reduces to the well-known Cauchy problem for an ordinary differential equation of order $n + 1$ with constant coefficients, which was studied in [4]. First, we shall give a result about the particular solution of the equation.

PROPOSITION 4.3. *Let h_u be as above. Then*

$$\lim_{t\to 0}\left(D_{\mu-q-1}D_\mu^l h_u * f\right)(t) = 0, \tag{4.12}$$

for any f in $\mathscr{H}$ and $l = 0, 1, \dots, n$; $q = 0, 1, \dots, \eta - 1$.

Proof. We have

$$g_{j,a,k}(t) = \left\langle \frac{z^j}{(z^\mu - a)^{1+k}}, e^{zt} \right\rangle = \left\langle \frac{1}{(x - a_i)^{1+k}}, \left\langle \frac{z^j}{z^\mu - x}, e^{zt} \right\rangle_z \right\rangle_x,$$

that is,

$$g_{j,a,k}(t) = \left\langle \frac{1}{(x - a_i)^{1+k}}, g_{j,x,0}(t) \right\rangle. \tag{4.13}$$

From (4.13) and (3.14) it is easy to see that

$$(g_{i,a,k} * g_{j,b,m})(t) = \left\langle \frac{1}{(x-a)^{1+k}(x-b)^{1+m}}, g_{i+j,x,0}(t) \right\rangle.$$

Let f be an element of $\mathscr{H}$. Then

$$f(t) = \sum_{j=0}^{\eta-1} \sum_{(m,l)\in\mathscr{I}} c_{j,l,m} g_{j,d_m,l}(t), \qquad c_{j,l,m} \in \mathbb{C},$$

for some index set of the type $\mathscr{I} = \{(m, l) : 0 \le m \le s,\ 0 \le l \le l_m - 1\}$, or equivalently

$$\begin{aligned} f(t) &= \sum_{j=0}^{\eta-1} \sum_{(m,l)\in\mathscr{I}} c_{j,l,m} \left\langle \frac{1}{(x-d_m)^{1+l}}, g_{j,x,0}(t) \right\rangle \\ &= \sum_{j=0}^{\eta-1} e_j \left\langle \frac{p(x)}{v(x)}, g_{j,x,0}(t) \right\rangle, \end{aligned}$$

where p/v is an element of $\mathscr{R}$ and the e_j's are complex numbers. Besides,

$$h_u = \sum_{(i,k)\in\mathscr{I}} \alpha_{i,k} g_{0,a_i,k} = \left\langle \frac{1}{u(x)}, g_{0,x,0}(t) \right\rangle,$$

so that

$$\begin{aligned} h_u * f = \sum_{j=0}^{\eta-1} & \sum_{(i,k)\in\mathscr{I},(m,l)\in\mathscr{I}} \alpha_{i,k} c_{j,l,m} \\ & \times \left\langle \frac{1}{(x-a_i)^{1+k}(x-d_m)^{1+l}}, g_{j,x,0}(t) \right\rangle \end{aligned}$$

or

$$h_u * f = \sum_{j=0}^{\eta-1} e_j \left\langle \frac{p(x)}{u(x)v(x)}, g_{j,x,0}(t) \right\rangle. \tag{4.14}$$

Hence, using (4.10) we obtain

$$\begin{aligned} &\lim_{t\to 0} \left(D_{\mu-q-1} D_\mu^l h_u * f \right)(t) \\ &\quad = \lim_{t\to 0} D_{\mu-q-1} \sum_{j=0}^{\eta-1} e_j \left\langle \frac{p(x)}{u(x)v(x)}, x^l g_{j,x,0}(t) \right\rangle \\ &\quad = e_q \left\langle \frac{x^l p(x)}{u(x)v(x)}, 1 \right\rangle \\ &\quad = 0. \end{aligned}$$

The last equality follows from degree(uv) − degree$(x^l p) \ge 2$, since the sum of the residues of a rational function with this property is zero. ∎

We can write the solution (4.9) in a different and useful form. Let G be a function of the type

$$G(x,t) = \sum_{j=0}^{\eta-1} c_j g_{j,x,0}(t), \tag{4.15}$$

where the coefficients c_j are arbitrary complex numbers and x and t are real or complex variables. Then

$$D_\mu G(x,t) = xG(x,t), \tag{4.16}$$

since $D_\mu g_{j,x,0} = x g_{j,x,0}$ for $j = 0,1,\ldots,\eta-1$. So, $G(x,t)$ is a generating function for D_μ and hence the general solution of the equation $u(D_\mu)y = 0$ can be written as (see [6, Proposition 7.1])

$$y(t) = \left\langle \frac{p(x)}{u(x)}, G(x,t) \right\rangle, \qquad p \in \mathscr{P}_n. \tag{4.17}$$

Let $u_k(x)$ be the Horner polynomials defined in (2.20). In particular, we consider the solution

$$v(t) = \sum_{m=0}^{n} \sum_{j=0}^{\eta-1} d_{m,j} \left\langle \frac{u_{n-m}(x)}{u(x)}, g_{j,x,0}(t) \right\rangle. \tag{4.18}$$

Then

$$D_\mu^l v(t) = \sum_{m=0}^{n} \sum_{j=0}^{\eta-1} d_{m,j} \left\langle \frac{u_{n-m}(x)}{u(x)}, x^l g_{j,x,0}(t) \right\rangle,$$

and using (4.10) and the biorthogonality property (2.21) we get

$$\lim_{t\to 0} \left(D_{\mu-q-1} D_\mu^l v \right)(t) = \sum_{m=0}^{n} d_{m,q} \left\langle \frac{u_{n-m}(x)}{u(x)}, x^l \right\rangle = d_{l,q}. \tag{4.19}$$

In consequence $v(t)$ is the solution in the space $\mathscr{H}$ of the Cauchy problem corresponding to (4.11) with $f = 0$.

By the PFD (2.6) we also have

$$\frac{u_{n-m}(x)}{u(x)} = \sum_{(i,k)\in\mathscr{I}} \frac{b_{i,k,m}}{(x-a_i)^{1+k}},$$

where $b_{i,k,m} = \langle L_{i,m_i-1-k}, u_{n-m} \rangle$. So

$$\left\langle \frac{u_{n-m}(x)}{u(x)}, g_{j,x,0}(t) \right\rangle = \sum_{(i,k)\in\mathscr{I}} b_{i,k,m} \left\langle \frac{1}{(x-a_i)^{1+k}}, g_{j,x,0}(t) \right\rangle$$
$$= \sum_{(i,k)\in\mathscr{I}} b_{i,k,m} g_{j,a_i,k}(t)$$

and

$$v(t) = \sum_{m=0}^{n} \sum_{j=0}^{\eta-1} \sum_{i=0}^{r} \sum_{k=0}^{m_i-1} d_{m,j} b_{i,k,m} g_{j,a_i,k}(t). \tag{4.20}$$

Combining Proposition 4.3 and (4.19) we obtain the following.

PROPOSITION 4.4. *The unique solution of the Cauchy boundary-value problem* (4.11) *in the space* $\mathscr{H}$ *can be represented in the form*

$$g(t) = (h_u * f)(t) + v(t), \tag{4.21}$$

where the function $v(t)$ *is given by* (4.18) *or* (4.20).

We know that the differential operators D_μ^i and $D_{i\mu}$ are different for $i = 2, 3, \dots, n+1$. However, from (4.14) we have

$$D_\mu^i (h_u * f)(t) = \sum_{j=0}^{\eta-1} e_j \left\langle \frac{x^i p(x)}{u(x)v(x)}, g_{j,x,0}(t) \right\rangle,$$

and a direct calculation yields

$$D_{i\mu} g_{j,x,0}(t) = \sum_{m=0}^{i-2} f_{i,j,m}(t) x^m + x^i g_{j,x,0}(t),$$

where

$$f_{i,j,m}(t) = \frac{t^{(m-i+1)\mu - j - 1}}{\Gamma(\mu(m-i+1)-j)},$$

thus using (4.14) again we get

$$D_{i\mu}(h_u * f)(t) = \sum_{j=0}^{\eta-1} \sum_{m=0}^{i-2} e_j f_{i,j,m}(t) \left\langle \frac{x^m p(x)}{u(x)v(x)}, 1 \right\rangle$$
$$+ D_\mu^i (h_u * f)(t).$$

But

$$\left\langle \frac{x^m p(x)}{u(x)v(x)}, 1 \right\rangle = 0,$$

for $m = 0, 1, \ldots, i - 2$; $i = 2, 3, \ldots, n + 1$. Therefore

$$D_\mu^i (h_u * f)(t) = D_{i\mu}(h_u * f)(t) \tag{4.22}$$

and we obtain the following corollary.

COROLLARY 4.2. *Let f be a function in $\mathscr{H}$. The unique solution of the boundary-value problem*

$$\sum_{i=0}^{n+1} b_i \big(D_{(n+1-i)\mu} y\big)(t) = f(t), \qquad \lim_{t\to 0} \big(D_{l\mu+\mu-q-1} y\big)(t) = 0, \tag{4.23}$$

for $l = 0, 1, \ldots, n$ and $q = 0, 1, \ldots, \eta - 1$, can be expressed as

$$y(t) = (h_u * f)(t), \tag{4.24}$$

where $b_0 = 1$ and u and h_u are defined in (2.1) *and* (3.23), *respectively.*

We will solve next two problems taken from [10]. We shall suppose that the function f in the right-hand side of the equation belongs to $\mathscr{H}$ in both examples.

EXAMPLE 1. For α in $\mathbb{C}$ let us consider the following Cauchy problem:

$$\big(D_\mu y\big)(t) - \alpha y(t) = f(t), \qquad \lim_{t\to 0} \big(D_{\mu-q-1} y\big)(t) = d_q, \tag{4.25}$$

where the d_q are arbitrary complex numbers and $q = 0, 1, \ldots, \eta - 1$. We apply Proposition 4.4 with $u(z) = z - \alpha$. In this case

$$h_u(t) = g_{0,\alpha,0}(t) = t^{\mu-1} E_{\mu,\mu}(\alpha t^\mu),$$

and using (4.20) we can write the solution of (4.25) as

$$y(t) = (g_{0,\alpha,0} * f)(t) + \sum_{j=0}^{\eta-1} d_j g_{j,\alpha,0}(t).$$

EXAMPLE 2. For the problem

$$\big(D_{3/2} y\big)(t) - \alpha y'(t) + \beta^2 \big(D_{1/2} y\big)(t) - \alpha\beta y(x) = f(t),$$

$$\alpha, \beta \in \mathbb{C}, \tag{4.26}$$

$$\lim_{t\to 0} \big(D_{-1/2}\big) y(t) = \lim_{t\to 0} y(t) = \lim_{t\to 0} \big(D_{1/2}\big) y(t) = 0,$$

we can use Corollary 4.2 with $u(z) = z^3 - \alpha z^2 + \beta^2 z - \alpha\beta^2$ and $\mu = 1/2$ to find its solution. We have that

$$\frac{1}{u(z)} = \frac{A}{z-\alpha} + \frac{B}{z - i\beta} + \frac{C}{z + i\beta},$$

where $A = 1/(\alpha^2 + \beta^2)$, $B = -1/2(\beta^2 + i\alpha\beta)$, $C = 1/2(-\beta^2 + i\alpha\beta)$. Thus

$$\begin{aligned} h_u &= Ag_{0,\alpha,0}(t) + Bg_{0,i\beta,0}(t) + Cg_{0,-i\beta,0}(t) \\ &= t^{-1/2}\big[AE_{1/2,1/2}(\alpha t^{1/2}) + BE_{1/2,1/2}(i\beta t^{1/2}) \\ &\qquad + CE_{1/2,1/2}(-i\beta t^{1/2})\big]. \end{aligned}$$

A direct calculation using (4.6) shows that

$$\begin{aligned} Bg_{0,i\beta,0}(t) + Cg_{0,-i\beta,0}(t) &= -A\left\langle \frac{z^{1/2}+\alpha}{z+\beta^2}, e^{zt}\right\rangle, \\ &= -A\big[t^{1/2}E_{1,1/2}(-\beta^2 t) + \alpha E_{1,1}(-\beta^2 t)\big], \end{aligned}$$

then, alternatively we have (see [10])

$$h_u = A\big[t^{-1/2}E_{1/2,1/2}(\alpha t^{1/2}) - t^{-1/2}E_{1,1/2}(-\beta^2 t) - \alpha E_{1,1}(-\beta^2 t)\big].$$

The solution is given by (4.24).

Finally, we observe that we can apply our results to find the solution of an equation with a more general function f in its right-hand side, provided that we have an integral representation for the convolution. See [10] and [14].

5. THE LAPLACE EQUATION

In this section we consider the differential equation

$$(a_2 t + b_2)y''(t) + (a_1 t + b_1)y'(t) + (a_0 t + b_0)y(t) = 0, \quad (5.1)$$

where a_k, b_k are given complex numbers for $k = 0,1,2$ and $a_2 \neq 0$. It is called the Laplace Equation in honor of Pierre Simon de Laplace (1749–1827), who studied it in his treatise "*Theórie analityque des probabilités*" of 1817.

Without loss of generality we can suppose that $b_2 = 0$ by taking $t + b_2/a_2$ as a new variable. Thus we shall discuss the differential equation

$$a_2 t y''(t) + (a_1 t + b_1) y'(t) + (a_0 t + b_0) y(t) = 0, \tag{5.2}$$

with $a_2 \neq 0$, in the domain $t \geq 0$.

Let P and Q be the polynomials defined by $P(z) = b_1 z + b_0$ and $Q(z) = a_2 z^2 + a_1 z + a_0$, and let L be the second-order differential operator defined by $L = P(D) + tQ(D)$. We can write Eq. (5.2) as

$$Ly = [P(D) + tQ(D)]y = 0. \tag{5.3}$$

We seek a solution of (5.3) of the form

$$g(t) = \langle h(z), e^{zt} \rangle \tag{5.4}$$

for an appropriate function h. First, note that

$$\begin{aligned} Lg &= \langle h(z), [P(z) + tQ(z)]e^{tz} \rangle \\ &= \langle h(z), [P(z) + Q(z)D_z]e^{tz} \rangle \\ &= \langle -D_z Q(z)h(z) + P(z)h(z), e^{tz} \rangle. \end{aligned}$$

Then, the function g given in (5.4) is a solution of (5.3) if h is a solution of the first-order differential equation

$$-D_z Q(z)h(z) + P(z)h(z) = 0,$$

or equivalently

$$D_z h(z) + \frac{Q'(z) - P(z)}{Q(z)} h(z) = 0. \tag{5.5}$$

Solving (5.5) we get

$$h(z) = Ce^{-H(z)}, \tag{5.6}$$

where C is an arbitrary constant and H is a function such that

$$H'(z) = \frac{Q'(z) - P(z)}{Q(z)} = -\frac{(-2a_2 + b_1)z - a_1 + b_0}{a_2 z^2 + a_1 z + a_0}.$$

The function H can be found by simple integration. We have to consider two cases.

THEOREM 5.1. *If the equation $Q(z) = 0$ has two distinct roots z_1 and z_2, then the function*

$$g(t) = \left\langle K(z - z_1)^A (z - z_2)^B, e^{zt} \right\rangle \tag{5.7}$$

is a solution of the equation (5.2) *for any constant K, where A and B are determined by the partial functions decomposition*

$$\frac{(-2a_2 + b_1)z - a_1 + b_0}{a_2 z^2 + a_1 z + a_0} = \frac{A}{z - z_1} + \frac{B}{z - z_2}.$$

Proof. In this case we have

$$H'(z) = -\frac{A}{z - z_1} - \frac{B}{z - z_2},$$

so that

$$H(z) = -\ln\left| K(z - z_1)^A (z - z_2)^B \right|,$$

where K is an arbitrary constant. Using this expression together with (5.4) and (5.6) we get (5.7). ∎

Note that we may use Proposition 3.2 to find a series representation of the solution (5.7). The following equations

$$t^2 y'' + ty' + (t^2 - \alpha^2) y = 0, \qquad \alpha \in \mathbb{C},$$
$$ty'' + (c - t) y' - ay = 0, \qquad a, c \in \mathbb{C},$$
$$ty'' - (t + \alpha - 1) y' + (\alpha + \lambda) y = 0, \qquad \alpha, \lambda \in \mathbb{C},$$

are the Bessel, confluent Hypergeometric, and Laguerre differential equations, respectively, and all them can be solved by this method. See [8].

For the second case we have the following.

THEOREM 5.2. *If the equation $Q(z) = 0$ has a double root a, then the function*

$$g(t) = K \sum_{k=0}^{\infty} \frac{(-1)^k B^k}{k!\Gamma(k - A)} t^{k-A-1} \tag{5.8}$$

is a solution of the equation (5.2) *for any constant K, where A and B are determined by the partial fractions decomposition*

$$\frac{(-2a_2 + b_1)z - a_1 + b_0}{a_2 z^2 + a_1 z + a_0} = \frac{A}{z - a} + \frac{B}{(z - a)^2}.$$

Proof. Now we have

$$H(z) = -\ln\left|C_1(z-a)^A\right| + \frac{B}{z-a}, \qquad C_1 \in \mathbb{C},$$

so that

$$h(z) = Ke^{-B/(z-a)}(z-a)^A, \qquad K \in \mathbb{C}.$$

Expanding h in a series of powers of $z - a$ we get

$$h(z) = K \sum_{k=0}^{\infty} \frac{(-1)^k B^k}{k!(z-a)^{k-A}},$$

and using (5.4) and (3.2) we get (5.8). ∎

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